J. Engg. Math., 16, 223–243 (1982). 0022–0083/82/03/0223–21 $00.20/0 


# Hodograph transformations and solutions in variably inclined MHD plane flows

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(Received June 30, 1981)

SUMMARY

By a variably inclined MHD plane flow we mean a flow in which the magnetic and velocity fields are coplanar, the angle between these vector fields is variable and all the flow variables are functions of two coordinates and time. No work seems to have been done for these general plane MHD flows, even in the steady case.

In the present paper the work in steady, viscous, incompressible flows is extended to general variably inclined, but nowhere aligned, flows with the objective of obtaining some exact solutions. We employ the hodograph transformation, one of the strong analytical methods, to find these solutions.

## 1. Introduction

A vast amount of research has been carried out on the motion of electrically conducting fluids, moving in a magnetic field, since Alfven's [1] classic work. Mathematical complexity of the phenomenon induced many researchers to adopt a rather useful alternate technique of investigating special classes of flows such as aligned or parallel flows, crossed or orthogonal flows, constantly inclined flows and transverse flows. These special classes of flows yielded various solvable second order mathematical structures and, furthermore, these structures aided in the determination of similarities and contrasts with ordinary fluid-dynamics. These results were often achieved by employing well established fluid-dynamical techniques. For example, in the case of an inviscid incompressible fluid in steady flow, Ladikov [2] has derived two Bernoulli type equations for orthogonal flows, Kingston and Talbot [3] have classified all orthogonal flows as radial, vortex, rectilinear or as certain types of spirals, Chandna and Nath [4] have established uniqueness properties for aligned flows and Waterhouse and Kingston [5] have determined all possible flow configurations for the constantly inclined case. Chandna et al. [6–13] have published a series of results for the steady viscous incompressible flow problem, obtaining various flow properties, geometries and solutions in the aligned, orthogonal and constantly inclined cases.

An excellent survey of this, with applications to numerous non-linear problems, has been given by Ames [14]. In the study of MHD, this method has been previously applied to aligned compressible flows by Smith [15], to viscous incompressible orthogonal flows by Chandna and Garg [12] and to constantly inclined incompressible flow by Barron and Chandna [13].

The plan of the paper is as follows. In Section 2 the basic flow equations are cast into convenient form for this work. Section 3 contains the transformation of equations to the hodograph plane. Under these transformations, it is determined that these flows are governed by a system of two partial differential equations in the Legendre transform function and the transformed variable angle between the vector fields. Theoretical development of this section is illustrated by solutions to the following examples in Section 4: i) a vortex flow problem with spiral magnetic lines; (ii) a radial flow problem with spiral magnetic lines; iii) hyperbolic flows with straight magnetic lines; and iv) spiral inviscid flows with spiral magnetic lines.

## 2. Equations of motion

The steady, plane flow of a viscous, incompressible fluid of infinite electrical conductivity is governed by the following system of equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{\partial p}{\partial x}=\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\mu H_2\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right),$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{\partial p}{\partial y}=\eta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\mu H_1\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right),$$

$$uH_2-vH_1=k,\qquad \frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0$$

where $u$, $v$ are the components of the velocity field $\mathbf{V}$, $H_1$, $H_2$ the components of the magnetic vector field $\mathbf{H}$ and $p$ is the pressure function; all being functions of $x, y$. In this system $\rho$, $\eta$, $\mu$, are respectively the constant fluid density, the constant coefficient of viscosity and the constant magnetic permeability. Furthermore, $k$ is an arbitrary constant of integration obtained from the diffusion equation curl $(\mathbf{V}\times\mathbf{H})=\mathbf{0}$; $k$ is zero for aligned flows and non-zero in the case of non-aligned flows.

Introducing the functions

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},\qquad j=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y},\qquad h=\tfrac{1}{2}\rho q^2+p, \tag{1}$$

where $q^2=u^2+v^2$, the above system of equations is replaced by the following system:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \text{(continuity)}$$

$$
\left.\begin{aligned}
\eta \frac{\partial \omega}{\partial y} - \rho v\omega + \mu j H_2 &= -\frac{\partial h}{\partial x}, \\
\eta \frac{\partial \omega}{\partial x} - \rho u\omega + \mu j H_1 &= \frac{\partial h}{\partial y},
\end{aligned}\right\} \quad \text{(linear momentum)}
$$

$$
\begin{aligned}
&uH_2 - vH_1 = k, && \text{(diffusion)} \qquad (2)\\
&\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0, && \text{(solenoidal)}\\
&\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega, && \text{(vorticity)}\\
&\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = j, && \text{(current density)}
\end{aligned}
$$

of seven non-linear partial differential equations in seven unknowns $u, v, H_1, H_2, \omega, j$ and $h$ as functions of $x, y$. This system has the advantage of being a system of first order. Martin [16] has successfully used such a reduction of order from two to one to study viscous non-MHD flows.

We now consider variably inclined plane flows and let $\alpha = \alpha(x, y)$ be this variable angle such that $\alpha(x, y) \neq 0$ for every $(x, y)$ in the flow region. The vector and scalar products of **V** and **H**, using the diffusion equation in (2), yield

$$
\begin{aligned}
uH_2 - vH_1 &= qH \sin\alpha = k, \\
uH_1 + vH_2 &= qH \cos\alpha = k \cot\alpha
\end{aligned} \qquad (3)
$$

where $H = \sqrt{H_1^2 + H_2^2}$. Considering these as two linear algebraic equations in the unknowns $H_1, H_2$, we solve these in terms of $u, v$ and $\alpha$; i.e.

$$
H_1 = \frac{k}{q^2}(u \cot\alpha - v), \qquad H_2 = \frac{k}{q^2}(v \cot\alpha + u). \qquad (4)
$$

Alternatively, one can solve (3) for $u, v$ in terms of $H_1, H_2$ and $\alpha$ to get

$$
u = \frac{k}{H^2}(H_1 \cot\alpha + H_2), \qquad v = \frac{k}{H^2}(H_2 \cot\alpha - H_1). \qquad (5)
$$

We now distinguish between two types of approaches. First, equations (4) can be employed to eliminate functions $H_1$ and $H_2$ from the system of equations (2). The unknown functions, to be determined, will then be $u$, $v$, $h$, $\omega$, $j$ and $\alpha$. Secondly, one can eliminate $u$ and $v$ from the system (2) by using equations (5). One then obtains a system of equations to be solved for $H_1$, $H_2$, $h$, $\omega$, $j$ and $\alpha$ as functions of $x, y$. The first approach leads us to the study of flows, after hodograph transformations, in the hodograph plane. Likewise, the second approach leads to the study in the magnetograph plane.

Taking the first approach, we eliminate functions $H_1$ and $H_2$ from the system of equations (2), by using equations (4), and obtain the following system of six partial differential equations:

$$
\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0,\\
&\eta\frac{\partial \omega}{\partial y}-\rho v\omega+\frac{\mu k}{q^2}(v\cot\alpha+u)j = -\frac{\partial h}{\partial x},\\
&\eta\frac{\partial \omega}{\partial x}-\rho u\omega+\frac{\mu k}{q^2}(u\cot\alpha-v)j = \frac{\partial h}{\partial y},\\
&(v^2-u^2-2uv\cot\alpha)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+(v^2\cot\alpha-u^2\cot\alpha+2uv)\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \qquad (6)\\
&\quad +q^2\left(u\frac{\partial}{\partial x}\cot\alpha+v\frac{\partial}{\partial y}\cot\alpha\right) = 0, \qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} = \omega,\\
&k\frac{\partial}{\partial x}\left(\frac{v\cot\alpha+u}{q^2}\right)-k\frac{\partial}{\partial y}\left(\frac{u\cot\alpha-v}{q^2}\right) = j
\end{aligned}
$$

for the six unknown functions $u$, $v$, $j$, $\omega$, $h$, $\alpha$ of $x$, $y$. Once a solution for this system is determined, the pressure function and the magnetic vector function are obtained by using the definition of $h$ in (1) and equations (4) respectively.

Taking the second approach, we eliminate functions $u$ and $v$ from the system of equations (2), by using equations (5), and obtain the following system of six partial differential equations:

$$
\begin{aligned}
&\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y} = 0,\\
&\eta\frac{\partial \omega}{\partial x}+\mu H_2 j-\frac{\rho k}{H^2}(H_2\cot\alpha-H_1)\omega = -\frac{\partial h}{\partial x},\\
&\eta\frac{\partial \omega}{\partial x}+\mu H_1 j-\frac{\rho k}{H^2}(H_1\cot\alpha+H_2)\omega = \frac{\partial h}{\partial y},\\
&(H_1^2-H_2^2-2H_1H_2\cot\alpha)\left(\frac{\partial H_1}{\partial y}+\frac{\partial H_2}{\partial x}\right)+(H_2^2\cot\alpha-H_1^2\cot\alpha-2H_1H_2) \qquad (7)\\
&\left(\frac{\partial H_1}{\partial x}-\frac{\partial H_2}{\partial y}\right)+H^2\left(H_1\frac{\partial}{\partial x}\cot\alpha+H_2\frac{\partial}{\partial y}\cot\alpha\right) = 0,\\
&k\frac{\partial}{\partial x}\left(\frac{H_2\cot\alpha-H_1}{H^2}\right)-k\frac{\partial}{\partial y}\left(\frac{H_1\cot\alpha+H_2}{H^2}\right) = \omega, \qquad \frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y} = j
\end{aligned}
$$

for the six unknown functions $H_1, H_2, j, \omega, h, \alpha$ of $(x,y)$. Once a solution of this system is determined, the velocity vector field is obtained from (5), and then the pressure function is found by using the definition of $h$ in (1).

## 3. Study of flows in the hodograph plane

Here we develop the first approach. System (6) of six partial differential equations is our starting point.

Letting the functions $u = u(x,y), v = v(x,y)$ to be such that, in the region of flow, the Jacobian

$$\bar{J}(x,y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0, \qquad |\bar{J}| < \infty, \tag{8}$$

we may consider $x$ and $y$ as functions of $u$ and $v$. By means of $x = x(u,v), y = y(u,v)$, we have the relations

$$\frac{\partial u}{\partial x} = \bar{J}\,\frac{\partial y}{\partial v}, \qquad \frac{\partial u}{\partial y} = -\bar{J}\,\frac{\partial x}{\partial v}, \qquad \frac{\partial v}{\partial x} = -\bar{J}\,\frac{\partial y}{\partial u}, \qquad \frac{\partial v}{\partial y} = \bar{J}\,\frac{\partial x}{\partial u}. \tag{9}$$

Furthermore, using (9), we have

$$\bar{J}(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} = J(u,v) \tag{10}$$

and

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial(f,y)}{\partial(x,y)} = \bar{J}\,\frac{\partial(f,y)}{\partial(u,v)} = J\,\frac{\partial(f,y)}{\partial(u,v)}, \\ \frac{\partial f}{\partial y} &= -\frac{\partial(f,x)}{\partial(x,y)} = \bar{J}\,\frac{\partial(x,f)}{\partial(u,v)} = J\,\frac{\partial(x,f)}{\partial(u,v)} \end{aligned} \tag{11}$$

where $f = f(x,y)$ is any continuously differentiable function and $f(u,v)$ is its transformed function in the $(u,v)$-plane.

Employing these transformation relations for the first order partial derivatives and equation (10) in the system of equations (6), the transformed system of partial differential equations in the $(u,v)$-plane is:

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0, \tag{12}$$

$$\eta J\,\frac{\partial(x,\omega)}{\partial(u,v)} - \rho v\omega + \frac{\mu k}{q^2}\,(v\cot\alpha + u)j \;=\; -J\,\frac{\partial(h,y)}{\partial(u,v)}, \tag{13}$$

$$\eta J\,\frac{\partial(\omega,y)}{\partial(u,v)} - \rho u\omega + \frac{\mu k}{q^2}\,(u\cot\alpha - v)j \;=\; J\,\frac{\partial(x,h)}{\partial(u,v)}, \tag{14}$$

$$\begin{aligned}&\left\{v(u^2+v^2)\frac{\partial}{\partial v}\cot\alpha - 2uv + (u^2-v^2)\cot\alpha\right\}\frac{\partial x}{\partial u}\\ &+\left\{2uv\cot\alpha + (u^2-v^2) - v(u^2+v^2)\frac{\partial}{\partial u}\cot\alpha\right\}\frac{\partial x}{\partial v}\\ &+\left\{2uv\cot\alpha + (u^2-v^2) - u(u^2+v^2)\frac{\partial}{\partial v}\cot\alpha\right\}\frac{\partial y}{\partial u}\\ &+\left\{u(u^2+v^2)\frac{\partial}{\partial u}\cot\alpha + 2uv + (v^2-u^2)\cot\alpha\right\}\frac{\partial y}{\partial v} \;=\; 0,\end{aligned} \tag{15}$$

$$J\left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}\right) \;=\; \omega, \tag{16}$$

$$kJ\left\{\frac{\partial\left(\dfrac{v\cot\alpha+u}{q^2},y\right)}{\partial(u,v)} - \frac{\partial\left(x,\dfrac{u\cot\alpha-v}{q^2}\right)}{\partial(u,v)}\right\} \;=\; j. \tag{17}$$

This is a system of six partial differential equations in six unknown functions $x(u,v)$, $y(u,v)$ and the four transformed functions $\omega(u,v)$, $h(u,v)$, $j(u,v)$, $\alpha(u,v)$ when $J = (\partial(x,y)/\partial(u,v))^{-1}$ is employed from (10). Once a solution set $x = x(u,v)$, $y = y(u,v)$, $\omega = \omega(u,v)$, $h = h(u,v)$, $j = j(u,v)$, $\alpha = \alpha(u,v)$ is determined for this system, we are lead to the solutions $u = u(x,y)$, $v = v(x,y)$ and, therefore, $\omega = \omega(x,y)$, $h = h(x,y)$, $j = j(x,y)$, $\alpha = \alpha(x,y)$ for the previous system of equations (6). The equation of continuity implies the existence of a streamfunction $\psi(x,y)$ so that

$$d\psi \;=\; -v\,dx + u\,dy, \qquad \text{or} \qquad \frac{\partial\psi}{\partial x} \;=\; -v, \qquad \frac{\partial\psi}{\partial y} \;=\; u. \tag{18}$$

Likewise, equation (12) implies the existence of a function $L(u,v)$, called the Legendre transform function of the streamfunction $\psi(x,y)$, so that

$$dL \;=\; -y\,du + x\,dv, \qquad \text{or} \qquad \frac{\partial L}{\partial u} \;=\; -y, \qquad \frac{\partial L}{\partial v} \;=\; x \tag{19}$$

and the two functions $\psi(x,y), L(u,v)$ are related by

$$L(u,v) = vx - uy + \psi(x,y). \tag{20}$$

Introducing $L(u,v)$ into the system (12) to (17), with $J$ given by (10), it follows that equation (12) is identically satisfied and this system may be replaced by

$$\eta J \frac{\partial\left(\frac{\partial L}{\partial v}, \omega\right)}{\partial(u,v)} - \rho v\omega + \frac{\mu k}{q^2}(v \cot\alpha + u)j = J\frac{\partial\left(h, \frac{\partial L}{\partial u}\right)}{\partial(u,v)}, \tag{21}$$

$$\eta J \frac{\partial\left(\omega, \frac{\partial L}{\partial u}\right)}{\partial(u,v)} + \rho u\omega - \frac{\mu k}{q^2}(u \cot\alpha - v)j = -J\frac{\partial\left(\frac{\partial L}{\partial v}, h\right)}{\partial(u,v)}, \tag{22}$$

$$\begin{aligned}&\left\{v^2 - u^2 - 2uv\cot\alpha + u(u^2+v^2)\frac{\partial}{\partial v}\cot\alpha\right\}\frac{\partial^2 L}{\partial u^2}\\ &\quad + \left\{2(u^2-v^2)\cot\alpha - 4uv - u(u^2+v^2)\frac{\partial}{\partial u}\cot\alpha + v(u^2+v^2)\frac{\partial}{\partial v}\cot\alpha\right\}\frac{\partial^2 L}{\partial u\,\partial v}\\ &\quad + \left\{u^2 - v^2 + 2uv\cot\alpha - v(u^2+v^2)\frac{\partial}{\partial u}\cot\alpha\right\}\frac{\partial^2 L}{\partial v^2} = 0,\end{aligned} \tag{23}$$

$$J\left(\frac{\partial^2 L}{\partial u^2} + \frac{\partial^2 L}{\partial v^2}\right) = \omega, \tag{24}$$

$$kJ\left\{\frac{\partial\left(\frac{\partial L}{\partial u}, \frac{v\cot\alpha + u}{q^2}\right)}{\partial(u,v)} + \frac{\partial\left(\frac{\partial L}{\partial v}, \frac{v - u\cot\alpha}{q^2}\right)}{\partial(u,v)}\right\} = j \tag{25}$$

with

$$\left[\frac{\partial^2 L}{\partial u^2}\frac{\partial^2 L}{\partial v^2} - \left(\frac{\partial^2 L}{\partial u\,\partial v}\right)^2\right]^{-1} = J \tag{26}$$

for the six functions $L(u,v)$, $h(u,v)$, $\omega(u,v)$, $j(u,v)$, $\alpha(u,v)$ and $J(u,v)$.

We now define

$$Q_1(u,v) = \frac{\partial\left(\frac{\partial L}{\partial v}, \omega\right)}{\partial(u,v)} = \frac{\partial\left(\frac{\partial L}{\partial v}, J\frac{\partial^2 L}{\partial u^2} + J\frac{\partial^2 L}{\partial v^2}\right)}{\partial(u,v)}, \tag{27}$$

$$Q_2(u,v) = \frac{\partial\left(\dfrac{\partial L}{\partial u}, \omega\right)}{\partial(u,v)} = \frac{\partial\left(\dfrac{\partial L}{\partial u}, J\dfrac{\partial^2 L}{\partial u^2} + J\dfrac{\partial^2 L}{\partial v^2}\right)}{\partial(u,v)}, \tag{28}$$

and use the integrability condition

$$\left(J\frac{\partial^2 L}{\partial u \partial v}\frac{\partial}{\partial v} - J\frac{\partial^2 L}{\partial v^2}\frac{\partial}{\partial u}\right)\left[J\frac{\partial\left(\dfrac{\partial L}{\partial u}, h\right)}{\partial(u,v)},\right]$$

$$= \left(J\frac{\partial^2 L}{\partial u^2}\frac{\partial}{\partial v} - J\frac{\partial^2 L}{\partial u \partial v}\frac{\partial}{\partial u}\right)\left[J\frac{\partial\left(\dfrac{\partial L}{\partial v}, h\right)}{\partial(u,v)}\right],$$

i.e. $\partial^2 h/\partial x \partial y = \partial^2 h/\partial y \partial x$, to eliminate $h(u,v)$ from equations (21) and (22) to obtain

$$\eta\left[\frac{\partial\left(\dfrac{\partial L}{\partial v}, JQ_1\right)}{\partial(u,v)} + \frac{\partial\left(\dfrac{\partial L}{\partial u}, JQ_2\right)}{\partial(u,v)}\right] - \rho(vQ_1 + uQ_2)$$

$$+ \frac{\mu k}{q^2}\left[(v\cot\alpha + u)\frac{\partial\left(\dfrac{\partial L}{\partial v}, j\right)}{\partial(u,v)} + (u\cot\alpha - v)\frac{\partial\left(\dfrac{\partial L}{\partial u}, j\right)}{\partial(u,v)}\right] = 0. \tag{29}$$

Equations (23) and (29) constitute a system of two non-linear partial differential equations in two unknowns $L(u,v)$, $\alpha(u,v)$ in the hodograph plane after $j$, $J$, $Q_1$, $Q_2$ are eliminated from (29) by using their differential expressions from (25) to (28). Therefore, a variably inclined steady plane flow of a viscous or inviscid fluid of infinite electrical conductivity must satisfy this system. Once a solution $L = L(u,v)$, $\alpha = \alpha(u,v)$ of this system is at hand, for which $J$ evaluated from (26) satisfies $0 < |J| < \infty$, the solutions for the velocity components $u(x,y)$, $v(x,y)$ are obtained by solving simultaneously equations (19) i.e. $x = \partial L/\partial v$, $y = -\partial L/\partial u$. Having obtained velocity components in the physical plane, we obtain $\alpha$ in the physical plane from the solution for $\alpha$ in the hodograph plane. Finally, $\omega(x,y)$, $h(x,y)$, $j(x,y)$, $H_1(x,y)$, $H_2(x,y)$ and $p(x,y)$ are determined by using system (6) and equations (4), (1).

We now derive equations (23), (29) in polar coordinates $(q,\theta)$ in $(u,v)$-plane. We have

$$q = \sqrt{u^2+v^2}, \qquad \theta = \tan^{-1}\frac{v}{u} \qquad \text{or} \qquad u = q\cos\theta, \qquad v = q\sin\theta \tag{30}$$

and

$$\frac{\partial}{\partial u} = \cos\theta \frac{\partial}{\partial q} - \frac{\sin\theta}{q}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial q} + \frac{\cos\theta}{q}\frac{\partial}{\partial\theta}. \tag{31}$$

Defining $L^*(q,\theta)$, $\alpha^*(q,\theta)$, $\omega^*(q,\theta)$, $j^*(q,\theta)$, $J^*(q,\theta)$ to be respectively the Legendre transform, variable angle, vorticity, current density, Jacobian function in $(q,\theta)$-coordinates, and using (30), (31) and

$$\frac{\partial(F,G)}{\partial(u,v)} = \frac{\partial(F^*,G^*)}{\partial(q,\theta)}\cdot\frac{\partial(q,\theta)}{\partial(u,v)} = \frac{1}{q}\frac{\partial(F^*,G^*)}{\partial(q,\theta)} \tag{32}$$

where $F(u,v)=F^*(q,\theta)$, $G(u,v)=G^*(q,\theta)$ are continuously differentiable functions, we obtain that $L^*(q,\theta)$ and $\alpha^*(q,\theta)$ satisfy

$$\begin{aligned}&\left(1-\frac{\partial}{\partial\theta}\cot\alpha^*\right)\frac{\partial^2L^*}{\partial q^2}+\left(\frac{\partial}{\partial q}\cot\alpha-\frac{2}{q}\cot\alpha\right)\frac{\partial^2L^*}{\partial\theta\partial q}-\frac{1}{q^2}\frac{\partial^2L^*}{\partial\theta^2}-\frac{1}{q}\frac{\partial L^*}{\partial q}\\&\quad+\left(\frac{2}{q^2}\cot\alpha-\frac{1}{q}\frac{\partial}{\partial q}\cot\alpha\right)\frac{\partial L^*}{\partial\theta}=0,\end{aligned} \tag{33}$$

$$\begin{aligned}&\eta\left[\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q}+\dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta},J^*Q_1^*\right)}{\partial(q,\theta)}+\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q}-\dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta},J^*Q_2^*\right)}{\partial(q,\theta)}\right]\\&\quad-\rho q^2(\sin\theta\,Q_1^*+\cos\theta\,Q_2^*)+\frac{\mu k}{q}\Bigg[(\sin\theta\cot\alpha^*+\cos\theta)\\&\quad\times\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q}+\dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta},j^*\right)}{\partial(q,\theta)}+(\cos\theta\cot\alpha^*-\sin\theta)\\&\quad\times\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q}-\dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta},j^*\right)}{\partial(q,\theta)}\Bigg]=0,\end{aligned} \tag{34}$$

where

$$J^*(q,\theta)=\frac{q^4}{q^2\dfrac{\partial^2L^*}{\partial q^2}\left(q\dfrac{\partial L^*}{\partial q}+\dfrac{\partial^2L^*}{\partial\theta^2}\right)-\left(\dfrac{\partial L^*}{\partial\theta}-q\dfrac{\partial^2L^*}{\partial q\partial\theta}\right)^2} \tag{35}$$

$$j^*(q,\theta) = \frac{kJ^*}{q^2}\left[\cot\alpha^* \frac{\partial^2 L^*}{\partial q^2} + \left(\frac{\partial^2 L^*}{\partial \theta^2} + q\frac{\partial L^*}{\partial q}\right)\left\{\frac{\partial}{\partial q}\left(\frac{\cot\alpha^*}{q}\right)\right\}\right.$$
$$\left. + \left(2 - \frac{\partial}{\partial\theta}\cot\alpha^*\right)\left\{\frac{\partial}{\partial q}\left(\frac{1}{q}\frac{\partial L^*}{\partial\theta}\right)\right\}\right], \tag{36}$$

$$Q_1^*(q,\theta) = Q_1(u,v) = \frac{1}{q}\,\frac{\partial\left(\sin\theta\,\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta},\ \omega^*\right)}{\partial(q,\theta)}, \tag{37}$$

$$Q_2^*(q,\theta) = Q_2(u,v) = \frac{1}{q}\,\frac{\partial\left(\cos\theta\,\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta},\ \omega^*\right)}{\partial(q,\theta)} \tag{38}$$

and

$$\omega^*(q,\theta) = J^*\left(\frac{\partial^2 L^*}{\partial q^2} + \frac{1}{q^2}\frac{\partial^2 L^*}{\partial\theta^2} + \frac{1}{q}\frac{\partial L^*}{\partial q}\right). \tag{39}$$

Once a solution $L^* = L^*(q,\theta)$, $\alpha^* = \alpha^*(q,\theta)$ of the system of equations (33), (34) is determined, we employ

$$x = \sin\theta\frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial\theta}, \qquad y = \frac{\sin\theta}{q}\frac{\partial L^*}{\partial\theta} - \cos\theta\frac{\partial L^*}{\partial q} \tag{40}$$

and (30) to obtain $u = u(x,y)$, $v = v(x,y)$ in the physical plane. The remaining flow variables are then obtained, in the physical plane, by using the flow equations in the physical plane.

## 4. Solutions

The general solution set of either system of equations (23), (29) in $(u,v)$-coordinates *or* (33), (34) in $(q,\theta)$-ordinates seems to be impossible. We, therefore, examine some special forms of solutions as applications of these systems.

EXAMPLE I: Vortex flow. In this example, we wish to determine the solution of a flow problem when the Legendre transform function is of the form $L^*(q,\theta) = F(q)$ in $(q,\theta)$-coordinates *or* $L(u,v) = F(\sqrt{u^2+v^2})$ in $(u,v)$-coordinates in the hodograph plane.

Let us assume

$$L^*(q,\theta) = F(q) \tag{41}$$

to be the Legendre transform function for the system of equations (33), (34) such that $F'(q) \neq 0, F''(q) \neq 0$.

Using (41) in (33), we find that $\alpha^*(q, \theta)$ satisfies

$$qF''(q)\frac{\partial}{\partial\theta}(\cot\alpha^*) + F'(q) = qF''(q)$$

and, therefore, its integration yields

$$\cot\alpha^* = \left(1 - \frac{1}{q}\frac{F'(q)}{F''(q)}\right)\theta + G(q) \tag{42}$$

where $G(q)$ is an arbitrary function.

By using (41) and (42) in (35) to (39), the expressions for $J^*, j^*, \omega^*, Q_1^*$ and $Q_2^*$ are found to be

$$\begin{aligned}
J^* &= \frac{q}{F'(q)F''(q)}, \qquad j^* = A(q)\theta + B(q),\\
\omega^* &= \frac{qF''(q) + F'(q)}{F'(q)F''(q)}, \qquad Q_1^* = -\left(\frac{F'(q)\cos\theta}{q}\right)\left(\frac{qF''(q) + F'(q)}{F'(q)F''(q)}\right)',\\
Q_2^* &= \left(\frac{F'(q)\sin\theta}{q}\right)\left(\frac{qF''(q) + F'(q)}{F'(q)F''(q)}\right)',
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
A(q) &= \left(\frac{k}{qF'(q)F''(q)}\right)\left(F''(q) - \frac{3}{q}F'(q) + \frac{2F'^2(q)}{q^2F''(q)} + \frac{F'^2(q)F'''(q)}{qF''^2(q)}\right),\\
B(q) &= \left(\frac{k}{qF'(q)F''(q)}\right)\left(G(q)F''(q) - \frac{G(q)}{q}F'(q) + F'(q)G'(q)\right).
\end{aligned} \tag{44}$$

We now eliminate the functions $L^*, \alpha^*, J^*, j^*, Q_1^*$ and $Q_2^*$ from (34) by using the expressions for these functions from (41) to (44) and find that $F(q)$ and $G(q)$ must satisfy

$$\mu k^2\left[F'\frac{d}{dq}\left(\frac{qF'^2F''' + q^2F''^3 - 3qF'F''^2 + 2F'^2F''}{q^3F'F''^3}\right)\right.$$
$$\left. + \left(\frac{qF'^3F''' + 2F'^3F'' - q^2F'^2F''F''' - 5qF'^2F''^2 + 4q^2F'F''^3 - q^3F''^4}{q^4F'F''^3}\right)\right]\theta$$

$$+\left[F'\frac{d}{dq}\left(\frac{qF'G'+qF''G-F'G}{q^2F'F''}\right)-\eta q\frac{d}{dq}\left\{\frac{F'}{F''}\frac{d}{dq}\left(\frac{qF''+F'}{F'F''}\right)\right\}\right.$$

$$\left.+\mu k^2G\left(\frac{3qF'F''^2-q^2F''^3-2F'^2F''-qF'^2F'''}{q^3F'F''^2}\right)\right] = 0, \tag{45}$$

if $L^*(q,\theta)$ given by (41) and, therefore, $\alpha^*(q,\theta)$ given by (42) form a solution set of the system of equations (33), (34).

Since (45) holds true for all values of $\theta$, if follows that $F(q)$ and $G(q)$ must satisfy

$$F'\frac{d}{dq}\left(\frac{qF'^2F'''+q^2F''^3-3qF'F''^2+2F'^2F''}{q^3F'F''^3}\right)$$

$$+\left(\frac{qF'^3F'''+2F'^3F''-q^2F'^2F''F'''-5qF'^2F''^2+4q^2F'F''^3-q^3F''^4}{q^4F'F''^3}\right) = 0 \tag{46}$$

and

$$F'\frac{d}{dq}\left(\frac{qF'G'+qF''G-F'G}{q^2F'F''}\right)-\eta q\frac{d}{dq}\left\{\frac{F'}{F''}\frac{d}{dq}\left(\frac{qF''+F'}{F'F''}\right)\right\} \tag{47}$$

$$+\mu k^2G\left(\frac{3qF'F''^2-q^2F''^3-2F'^2F''-qF'^2F'''}{q^3F'F''^2}\right) = 0.$$

Any solution set of these two non-linear ordinary differential equations in $F(q)$, $G(q)$ leads us to the solution of a particular flow problem. In the following, we study one of these solutions sets.

A simple solution of (46) such that $F'(q)\neq 0$, $F''(q)\neq 0$ is

$$F(q) = M_1q^2+M_2 \tag{48}$$

where $M_1\neq 0$, $M_2$ are two arbitrary constants. Using (48) in (47) and solving the resulting differential equation, we find

$$G(q) = M_3q^2+M_4, \tag{49}$$

where $M_3, M_4$ are arbitrary constants. Employing (48), (49) in (41), (42), we find that

$$L^*(q,\theta) = M_1q^2+M_2, \qquad \text{or} \qquad L(u,v) = M_1(u^2+v^2)+M_2,$$

$$\alpha^*(q,\theta) = \cot^{-1}(M_3q^2+M_4), \qquad \text{or} \qquad \alpha(u,v) = \cot^{-1}(M_3u^2+M_3v^2+M_4), \tag{50}$$

is a solution set of the system of partial differential equations (33), (34) or (23), (29).

Using the Legendre transform function from (50) in equation (40) and (43), expressions for the velocity components, the vorticity and the current density are obtained as

$$u = \frac{-y}{2M_1}, \qquad v = \frac{x}{2M_1}, \qquad \omega = \frac{1}{M_1}, \qquad j = \frac{kM_3}{M_1}. \tag{51}$$

The variable angle between the velocity and magnetic fields, in the physical plane, is obtained by using (51) in (50)

$$\alpha(x,y) = \cot^{-1}\left[\frac{M_3(x^2+y^2)}{4M_1^2} + M_4\right] \tag{52}$$

and, therefore, the magnetic field, by using (51) and (52) in (4), is given by

$$H_1 = -k\left[\frac{2M_1(x+M_4y)}{(x^2+y^2)} + \frac{M_3}{2M_1}y\right], \qquad H_2 = k\left[\frac{2M_1(M_4x-y)}{(x^2+y^2)} + \frac{M_3x}{2M_1}\right]. \tag{53}$$

Finally, employing (51) to (53) in the linear momentum equations of system (2) and integrating, we find the function $h(x,y)$. Using this solution for $h(x,y)$ and (51) in the definition of $h(x,y)$ in (1), the pressure function is

$$\begin{aligned} p(x,y) = {} & \frac{\rho}{8M_1^2}(x^2+y^2) + 2M_3\mu k^2 \tan^{-1}\frac{x}{y} - \frac{M_3^2\mu k^2}{4M_1^2}(x^2+y^2) \\ & - M_3M_4\mu k^2 \ln(x^2+y^2) + M_5, \end{aligned} \tag{54}$$

where $M_5$ is an arbitrary constant.

Summing up: 'A variably inclined steady plane MHD flow problem with the families of streamlines and magnetic lines given by

$$x^2+y^2 = \text{constant},$$
$$M_3(x^2+y^2) + 4M_1^2M_4 \ln(x^2+y^2) + 8M_1^2 \tan^{-1}(y/x) = \text{constant}$$

has the solutions $u(x,y)$, $v(x,y)$, $H_1(x,y)$, $H_2(x,y)$ and $p(x,y)$ obtained in equations (51), (53) and (54).'

EXAMPLE II: Radial flow. In this example, we wish to determine the solution of a flow problem when the Legendre transform function is of the form $L^*(q,\theta) = F(\theta)$ in $(q,\theta)$-coordinates or $L(u,v) = F\{\tan^{-1}(v/u)\}$ in $(u,v)$-coordinates in the hodograph plane.

Let us assume that

$$L^*(q,\theta) = F(\theta) \tag{55}$$

is the Legendre transform for the system of equations (33), (34) such that $F'(\theta) \neq \theta$.

From (55) and (33), we have the partial differential equation

$$q\frac{\partial}{\partial q}(\cot\alpha^*) - 2\cot\alpha^* + \frac{F''(\theta)}{F'(\theta)} = 0 \tag{56}$$

satisfied by $\alpha^*(q,\theta)$. The general solution of (56) is

$$\cot\alpha^* = \frac{F''(\theta)}{2F'(\theta)} + q^2 G(\theta), \tag{57}$$

where $G(\theta)$ is an arbitrary function of $\theta$. Employing (55), (57) in (35) to (39), we have

$$\begin{aligned} J^*(q,\theta) &= -\frac{q^4}{F'^2(\theta)}, \qquad j^*(q,\theta) = A(\theta) + q^2 B(\theta), \\ \omega^*(q,\theta) &= -q^2\frac{F''(\theta)}{F'^2(\theta)}, \qquad Q_1^*(q,\theta) = \left(\frac{F'''(\theta)\cos\theta - 2F''(\theta)\sin\theta}{qF'(\theta)}\right) \end{aligned} \tag{58}$$

and

$$Q_2^*(q,\theta) = -\left(\frac{F'''(\theta)\sin\theta + 2F''(\theta)\cos\theta}{qF'(\theta)}\right),$$

where

$$\begin{aligned} A(\theta) &= k\left(\frac{4F'^2(\theta) + 2F''^2(\theta) - F'(\theta)F'''(\theta)}{2F'^3(\theta)}\right), \\ B(\theta) &= -k\left(\frac{F'(\theta)G'(\theta) + F''(\theta)G(\theta)}{F'^2(\theta)}\right). \end{aligned} \tag{59}$$

We now use (55), (57), (58), (59) in equation (34) and obtain that the functions $F(\theta), G(\theta)$ must satisfy

$$\begin{aligned} &(8\eta F'F'' + 2\eta F'F^{iv} + 4\rho F'^2F'' - 4\mu k^2 F'^3 GG' - 4\mu k^2 F'^2 F'' G^2)q^4 \\ &\quad + 2\mu k^2(F'^3G'' + F'^2F''G' + F'^2F'''G - F'F''^2G)q^2 \\ &\quad + \mu k^2(4F'^2F'' + 6F''^3 - 6F'F''F''' + F'^2F^{iv}) = 0, \end{aligned} \tag{60}$$

if the assumed Legendre transform function in (55) and, thereby, the derived variable angle in (57) form a solution set of equations (33), (34). Since equation (60) holds true for all values of $q$, it follows that $F(\theta)$ and $G(\theta)$ must satisfy

$$F'^2F^{iv} - 6F'F''F''' + GF''^3 + 4F'^2F'' = 0, \tag{61}$$

$$F'^3G'' + F'^2F''G' + (F'^2F''' - F'F''^2)G = 0, \tag{62}$$

$$2\eta F'F^{iv} + 8\eta F'F'' + 4\rho F'^2F'' - 4\mu k^2F'^3GG' - 4\mu k^2F'^2F''G^2 = 0. \tag{63}$$

Every solution set $\{F(\theta), G(\theta)\}$ of these three non-linear ordinary differential equations leads us to the solution of a particular flow problem. In the following, we study one of the flow problems.

A simple solution of (61) such that $F'(\theta) \neq 0$ is

$$F(\theta) = N_1\theta + N_2, \tag{64}$$

where $N_1 \neq 0$, $N_2$ are arbitrary constants. This solution for $F(\theta)$ and equations (62), (63) require that $G(\theta)$ must satisfy $G''(\theta) = 0$ and $G(\theta)G'(\theta) = 0$, that is, $G(\theta)$ is an arbitrary constant. If this arbitrary constant is zero, then (64) and (57) imply that $\alpha^*(q, \theta) = \pi/2$, that is, the streamlines and the magnetic lines are everywhere orthogonal. This problem has been discussed by Chandna and Garg [12]. Here we study the case when $F(\theta)$ is given by (64), $G(\theta) = N_3, N_3 \neq 0$ is an arbitrary constant, and, therefore, have

$$\begin{aligned} L^*(q,\theta) &= N_1\theta + N_2 \quad \text{or} \quad L(u,v) = N_1 \tan^{-1}\frac{v}{u} + N_2, \\ \alpha^*(q,\theta) &= \cot^{-1}(N_3q^2) \quad \text{or} \quad \alpha(u,v) = \cot^{-1}(N_3u^2 + N_3v^2) \end{aligned} \tag{65}$$

as a solution set of the partial differential equations (33), (34) *or* (23), (29).

Following the previous example we find that

$$\begin{aligned} u(x,y) &= \frac{N_1x}{x^2+y^2}, \qquad v(x,y) = \frac{N_1y}{x^2+y^2}, \\ H_1(x,y) &= \frac{k}{N_1}\left[\frac{N_1^2N_3x - y(x^2+y^2)}{x^2+y^2}\right], \qquad H_2(x,y) = \frac{k}{N_1}\left[\frac{N_1^2N_3y + x(x^2+y^2)}{x^2+y^2}\right], \\ p(x,y) &= \frac{\mu k^2}{N_1^2}\left[2N_1^2N_3\tan^{-1}\frac{y}{x} - (x^2+y^2)\right] - \frac{\rho N_1^2}{2(x^2+y^2)} + N_4, \end{aligned} \tag{66}$$

where $N_4$ is an arbitrary constant and $N_2 > 0$.

Summing up: 'A variably inclined steady plane MHD flow problem with the families of streamlines and magnetic lines given by $y/x =$ constant,

$(x^2+y^2)-2N_1^2N_3\tan^{-1}(y/x) = $ constant, having the variable angle

$\alpha(x,y) = \cot^{-1}[N_3N_1^2/(x^2+y^2)]$ between them, has the solutions given by (66).'

EXAMPLE III: Hyperbolic flow. In this example, we take

$$L(u,v) = Au^2+Bv^2, \tag{67}$$

where $A$ and $B$ are two non-zero and un-equal real numbers and look for soluitons of a flow problem when the Legendre transform in $(u,v)$-coordinates in the hodograph plane is of this form, satisfying the system of equations (23), (29).

Using (67) in (23), the partial differential equation in $\alpha(u,v)$ is given by

$$Bv(u^2+v^2)\frac{\partial}{\partial u}(\cot\alpha)-Au(u^2+v^2)\frac{\partial}{\partial v}(\cot\alpha)+(2Auv-2Buv)(\cot\alpha)$$
$$+(B-A)(v^2-u^2) = 0. \tag{68}$$

The general solution of (68) is

$$\cot\alpha = \frac{(A-B)uv}{Au^2+Bv^2}+B(u^2+v^2)F(Au^2+Bv^2), \tag{69}$$

where $F(Au^2+Bv^2)$ is an arbitrary function of its argument.

Using expressions for $L(u,v), \alpha(u,v)$ from (67), (69) in equations (24) to (28), we have

$$J(u,v) = \frac{1}{4AB}, \qquad \omega = \frac{A+B}{2AB},$$

$$j(u,v) = k\left(\frac{A+B}{2A}\right)F(Au^2+Bv^2)+kB(u^2+v^2)F'(Au^2+Bv^2)+\frac{k(B-A)uv}{(Au^2+Bv^2)^2},$$

$$Q_1(u,v) = Q_2(u,v) = 0. \tag{70}$$

We now use (67), (69) and (70) in equation (29) and obtain the equation that $A, B$ and $F(Au^2+Bv^2)$ must satisfy so that the assumed $L(u,v)$ is the Legendre transform and derived $\alpha(u,v)$ is the transformed variable angle. This equation is

$$2AB(u^2+v^2)F''+2B(B-A)uvFF'+\left[\frac{(A+B)(Au^2+Bv^2)+2AB(u^2+v^2)}{Au^2+Bv^2}\right]F'$$

$$+(A-B)\left[\frac{Au^2-Bv^2}{(Au^2+Bv^2)^2}\right]F+\frac{2A(A-B)uv}{(Au^2+Bv^2)^3} = 0, \tag{71}$$

where $F'$ and $F''$ are the first and second derivatives of $F$ with respect to its argument. Any solution of this non-linear ordinary differential equation leads us to the solution of a particular flow problem. To obtain a solution of (71), we assume a solution of the form

$$F(Au^2 + Bv^2) = C(Au^2 + Bv^2)^m, \tag{72}$$

where $C, m$ are two arbitrary real numbers. Substituting (72) in (71), we have

$$\begin{aligned} &2C^2 mB(B-A)uv(Au^2 + Bv^2)^{2m-1} + [2Cm^2AB(u^2 + v^2) \\ &\quad + 2CB(B-A)v^2](Au^2 + Bv^2)^{m-2} + C(Am + Bm + A - B)(Au^2 + Bv^2)^{m-1} \\ &\quad + 2A(A-B)uv(Au^2 + Bv^2)^{-3} = 0. \end{aligned}$$

Since this equation is identically satisfied if and only if $m = -1$ and $C = \pm\sqrt{-A/B}$ or $C^2 = -A/B$, it follows that:

(i) one of the two unequal real numbers $A, B$ is to be a positive number and the other to be a negative number,

(ii) $F(Au^2 + Bv^2) = \pm\sqrt{-A/B}\,(Au^2 + Bv^2)^{-1}$

are the only solutions of (71) having the assumed form (72). Therefore, taking $C > 0$ and

$$A = a^2 > 0; \qquad B = -b^2 > 0; \qquad a, b \in R,$$

we have

$$L(u,v) = a^2u^2 - b^2v^2, \qquad \alpha(u,v) = \cot^{-1}\left[\frac{av - bu}{au + bv}\right] \tag{73}$$

to be a solution set of equations (23), (29). Following the previous examples we find that

$$\begin{aligned} &u(x,y) = -\frac{y}{2a^2}, \qquad v(x,y) = -\frac{x}{2b^2}, \\ &H_1(x,y) = \frac{2kab^2}{by + ax}, \qquad H_2(x,y) = -\frac{2ka^2b}{by + ax} \\ &p(x,y) = \frac{-\rho}{8a^2b^2}(x^2 + y^2) - \frac{2\mu k^2a^2b^2(a^2 + b^2)}{(by + ax)^2} + C, \end{aligned} \tag{74}$$

where $C$ is an arbitrary constant.

Summing up: 'A variably inclined steady plane MHD flow problem with the families of streamlines and magnetic lines given by $a^2x^2 - b^2y^2 = \text{constant}$, $by + ax = \text{constant}$, having the variable angle

$$\alpha = \cot^{-1}\left\{\frac{a^3x - b^3y}{ab(by + ax)}\right\}$$

between them, has solutions given by (74).'

Another variably inclined flow problem, when $C<0$ and $A=a^2, B=-b^2<0; a,b \in R$, corresponds to the solution set

$$L(u,v) = a^2u^2 - b^2v^2, \qquad \alpha(u,v) = \cot^{-1}\left(\frac{av+bu}{au-bv}\right) \tag{75}$$

of equations (23), (29). Using this solution set, we find that the streamlines $a^2x^2 - b^2y^2 =$ constant and mangetic lines $ax-by=$ constant form a variably inclined flow problem with

$$\alpha(x,y) = \cot^{-1}\left[\frac{a^3x+b^3y}{ab(by-ax)}\right].$$

Solutions to this problem are

$$\begin{aligned} u(x,y) &= \frac{-y}{2a^2}, \qquad v(x,y) = \frac{-x}{2b^2}, \\ H_1(x,y) &= \frac{2kab^2}{ax-by}, \qquad H_2(x,y) = \frac{2ka^2b}{ax-by}, \\ p(x,y) &= -\frac{\rho}{8a^2b^2}(x^2+y^2) - \frac{2\mu k^2a^2b^2(b^2+a^2)}{(ax-by)^2} + C, \end{aligned} \tag{76}$$

where $C$ is an arbitrary constant.

EXAMPLE IV: Spiral flow. We take the Legendre transform function, in $(q, \theta)$-coordinates, satisfying the system of equations (33), (34) to be

$$L^*(q,\theta) = A\theta + Bq^2 + C, \tag{77}$$

where $A \neq 0, B \neq 0$ and $C$ are real constants.

Using (77) in (33), it is found that $\alpha^*(q,\theta)$ satisfies the partial differential equation

$$Aq\frac{\partial}{\partial q}(\cot\alpha^*) + 2Bq^2\frac{\partial}{\partial\theta}(\cot\alpha^*) = 2A\cot\alpha^* = 0. \tag{78}$$

The general solution of (78) is

$$\cot\alpha^* = q^2F\left(\frac{B}{A}q^2 - \theta\right), \tag{79}$$

where $F$ is an arbitrary function of $(B/A)q^2 - \theta$.

Using the expressions for $L^*$, $\alpha^*$ from (77), (79) in (35) to (39), we have

$$
\begin{aligned}
J^*(q,\theta) &= \frac{q^4}{4B^2q^4 - A^2},\\
j^*(q,\theta) &= \left(\frac{kq^2}{4B^2q^4 - A^2}\right)\left[4Bq^2F + \left(\frac{4B^2q^4 - A^2}{A}\right)F' - \frac{2A}{q^2}\right],\\
Q_1^*(q,\theta) &= \frac{(16A^2Bq)(2Bq^2\cos\theta - A\sin\theta)}{(4B^2q^4 - A^2)^2},\\
Q_2^*(q,\theta) &= -\frac{(16A^2Bq)(2Bq^2\sin\theta + A\cos\theta)}{(4B^2q^4 - A^2)^2},\\
\omega^*(q,\theta) &= \frac{4Bq^4}{4B^2q^4 - A^2},
\end{aligned}
\tag{80}
$$

where $F'$ denotes differentiation of $F$ with respect to its argument. We use (77), (79), (80) in equation (34) to eliminate $L^*, \alpha^*, J^*, j^*, Q_1^*, Q_2^*$ and obtain the equation

$$
\begin{aligned}
&-\left(\frac{1024\mu k^2B^{10}}{A^2}F''\right)q^{20} + \left(\frac{512\mu k^2AB^8FF' - 2048\mu k^2B^9F'}{A}\right)q^{18}\\
&+ (1280\mu k^2B^8F'')q^{16} + (2048\mu k^2AB^7F' - 512\mu k^2A^2B^6FF')q^{14}\\
&+ (1024A^2B^5\eta + 256A^3B^5\rho - 640\mu k^2A^2B^6F'' - 256\mu k^2A^3B^5F^2\\
&+ 1024\mu k^2A^2B^6F - 1024\mu k^2AB^7)q^{12} + (192\mu k^2A^4B^4FF' - 768\mu k^2A^3B^5F')q^{10}\\
&+ (1024A^4B^3\eta - 128A^5B^3\rho + 160\mu k^2A^4B^4F'' + 128\mu k^2A^5B^3F^2\\
&- 512\mu k^2A^4B^4F + 512\mu k^2A^3B^5)q^8 + (128\mu k^2A^5B^3F' - 32\mu k^2A^6B^2FF')q^6\\
&+ (64A^6B\eta + 16A^7B\rho - 20\mu k^2A^6B^2F'' - 16\mu k^2A^7BF^2 + 64\mu k^2A^6B^2F\\
&- 64\mu k^2A^5B^3)q^4 + (2\mu k^2A^8FF' - 8\mu k^2A^7BF')q^2 + \mu k^2A^8F'' = 0
\end{aligned}
\tag{81}
$$

to be satisfied by $F\{(B/A)q^2 - \theta\}$ so that the assumed $L^*(q,\theta)$ and thereby the obtained $\alpha^*(q,\theta)$ form a solution set of equations (33), (34).

Equation (81) is a twentieth degree equation in $q$ and the coefficients of different powers of $q$ are functions of $(B/A)q^2 - \theta$. Since

$$
\left|\frac{\partial\left(\frac{B}{A}q^2 - \theta, q\right)}{\partial(q,\theta)}\right| = 1 \neq 0
$$

for every $(q, \theta)$, it follows that $(B/A)q^2 - \theta$ and $q$ may be considered as independent variables. Using this choice of independent variables and the fact that equation (81) holds true for different values of $q$, we are lead to the consequence that all the coefficients of different powers of $q$ in (81) must be zero. We, therefore, have the following equations to be satisfied by $F\{(B/A)q^2 - \theta)$:

$$F'' = 0, \tag{82}$$

$$AFF' - 4BF' = 0, \tag{83}$$

$$8A\eta + 2A^2\rho - 5\mu k^2ABF'' - 2\mu k^2A^2F^2 + 8\mu k^2ABF - 8\mu k^2B^2 = 0, \tag{84}$$

$$32A\eta + 4A^2\rho + 5\mu k^2ABF'' + 4\mu k^2A^2F^2 - 16\mu k^2ABF + 16\mu k^2B^2 = 0, \tag{85}$$

$$16A\eta + 4A^2\rho - 5\mu k^2ABF'' - 4\mu k^2A^2F^2 + 16\mu k^2ABF - 16\mu k^2B^2 = 0. \tag{86}$$

From (82) to (86), it follows that $F\{(B/A)q^2 - \theta\}$ must be such that either $F = 4B/A$ *or* $F$ is some other constant and, furthermore, $A$, $B$ and $F$ are related by

$$\begin{aligned} 4A\eta + A^2\rho - \mu k^2A^2F^2 + 4\mu k^2ABF - 4\mu k^2B^2 &= 0, \\ 8A\eta - A^2\rho + \mu k^2A^2F^2 - 4\mu k^2ABF + 4\mu k^2B^2 &= 0. \end{aligned} \tag{87}$$

From equation (87), we find that $\eta$ must be zero and

$$F = \frac{2B}{A} \pm \frac{1}{k}\sqrt{\frac{\rho}{\mu}}.$$

Therefore,

$$\begin{aligned} L^*(q, \theta) &= A\theta + Bq^2 + C, \\ \alpha^*(q, \theta) &= \cot^{-1}\left[\frac{2B}{A} \pm \frac{1}{k}\sqrt{\frac{\rho}{\mu}}\right]q^2 \end{aligned} \tag{88}$$

and

$$\begin{aligned} L^*(q, \theta) &= A\left(\theta \pm \frac{1}{2k}\sqrt{\frac{\rho}{\mu}}\,q^2\right) + C, \\ \alpha^*(q, \theta) &= \cot^{-1}\left(\frac{4B}{A}\,q^2\right) \end{aligned} \tag{89}$$

are solution sets of the system of equations (33), (34) with $\eta = 0$.

Following the previous examples, solutions to every variably inclined inviscid spiral flow resulting from these solution sets can be determined.

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